2050C Special Tutorial 3

4.1 and 4.2

- Definition. Let f be defined on $(a,b)\setminus\{x_0\}$ where $x_0\in(a,b)$. f converges to L if $\forall \varepsilon>0, \exists\delta$ s.t. $|f(x)-L|<\varepsilon$, $\forall x\in(a,b)$, $0<|x-x_0|<\delta$. (Here δ depends on ε, sometimes write δ_ε.)
- To compute limits we rely on
  - ~ Definition of Limit (ε-δ)
  - ~ Limit Theorem
  - ~ Sequential Criterion (usually used in proving divergence)

e.g. 1 Use ε-δ def. and Limit Theorem to prove

$$\lim_{x\to 1}\frac{x^2-2x+2}{3x-1}=\frac{1}{2}$$

Method I. $\dfrac{x^2-2x+2}{3x-1}-\dfrac{1}{2}=\dfrac{2x^2-7x+5}{2(3x-1)}=\dfrac{(2x-5)}{2(3x-1)}(x-1)$.

Frist, we pick $\delta_1=1/2$ and consider x, $|x-1|<\frac{1}{2}$. For then $1-\frac{1}{2}<x<1+\frac{1}{2}$, $\frac{1}{2}<x<\frac{3}{2}$, implies

$$0<\frac{2x-5}{2(3x-1)}\leq\frac{|2x|+5}{2(3x-1)}\leq\frac{8}{2(3/2-1)}=8.$$

therefore,

$$\left|\frac{x^2-2x+2}{3x-1}-\frac{1}{2}\right|\leq 8|x-1|,\quad \forall x, |x-1|<\frac{1}{2}.$$

Now, let $\delta=\min\{1/2, \varepsilon/8\}$, then for x, $|x-1|<\delta$,

$$\left|\frac{x^2-2x+2}{3x-1}-\frac{1}{2}\right|\leq 8|x-1|< 8\cdot\frac{\varepsilon}{8}=\varepsilon.\ \#$$

<u>Method II</u>. Using $\lim\limits_{x\to 1}(x^2-2x+2)=1$, $\lim\limits_{x\to 1}(3x-1)=2$

(For polynomial p, $\lim\limits_{x\to x_0} p(x)=p(x_0)$. You can use this fact.)

By Limit Theorem,

$$\lim_{x\to 1}\frac{x^2-2x+2}{3x-1}=\frac{\lim\limits_{x\to 1}(x^2-2x+2)}{\lim\limits_{x\to 1}(3x-1)}=\frac{1}{2}.$$

e.g. 2. Find $$\lim_{x\to 1}\frac{3x^2-3}{x^2-2x+1}.$$

So formally, $\lim\limits_{x\to 1}\frac{0}{0}$, no good. Need to observe for $x\neq 1$,

we have cancellation:

$$\frac{3x^2-3}{x^2+x-2}=\frac{3(x-1)(x+1)}{(x-1)(x+2)}=\frac{3(x+1)}{x+2}.$$

By Limit Theorem,

$$\lim_{x\to 1}\frac{3x^2-3}{x^2+x-2}=\lim_{x\to 1}\frac{3(x+1)}{x+2}$$

$$=\frac{\lim\limits_{x\to 1}3(x+1)}{\lim\limits_{x\to 1}x+2}$$

$$=\frac{6}{3}=2.$$

You can also use ε-δ to prove this.

e.g. 3 Study $$\lim_{x\to 0}\cos\frac{1}{x}.$$

We claim the limit doesn't exist. Let $x_n = 1/2n\pi$. then

$$\cos \frac{1}{x_n} = \cos 2n\pi = 1, \quad \forall n \geq 1.$$

On the other hand, let $y_n = \dfrac{1}{(2n+\frac{1}{2})\pi}$, $\cos y_n = 0$, $\forall n \geq 1$

Thus, we have $x_n \to 0$, $\cos \frac{1}{x_n} \to 1$

$y_n \to 0$, $\cos \frac{1}{y_n} \to 0$

By Sequential Criterion, the limit $\lim\limits_{x \to 0} \cos 1/x$ not exists.

Exercise

(1) Use ε-δ-def. and Limit Theorem to study

(a) $\lim\limits_{x \to 1} \dfrac{x^2+4x-4}{x+1}$ (b) $\lim\limits_{x \to 2} \dfrac{x^2-4}{3x-6}$

(2) Prove the limits do not exist:

(a) $\lim\limits_{x \to 0} \dfrac{x+2}{x^2+x^3}$, (b) $\lim\limits_{x \to 0} (x+1) \sin \dfrac{1}{x^2}$.

Hint: Use Sequential Criterion

Hint: use the fact that

$\lim\limits_{x \to x_0} f(x) = L$ implies

that $|f(x)| \leq L+1$ for

x, $0 < |x - x_0| < \delta$.

So, f is bdd near x_0.